Simplex Subdivisions and Nonnegativity Decision of Forms*

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Abstract This paper mainly studies nonnegativity decision of forms based on variable substitutions. Unlike existing research, the paper regards simplex subdivisions as new perspectives to study variable substitutions, gives some subdivisions of the simplex $\mathbb{T}_n$, introduces the concept of convergence of the subdivision sequence, and presents a sufficient and necessary condition for the convergent self-similar subdivision sequence. Then the relationships between subdivisions and their corresponding substitutions are established. Moreover, it is proven that if the form F is indefinite on $\mathbb{T}_n$ and the sequence of the successive L-substitution sets is convergent, then the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating, and an algorithm for deciding indefinite forms with a counter-example is obtained. Thus, various effective substitutions for deciding positive semi-definite forms and indefinite forms are gained, which are beyond the weighted difference substitutions characterized by "difference".

Key words Simplex Subdivisions; Nonnegativity decision of forms; The weighted difference substitutions

AMS subject classification(2000) 15A18, 65Y99

1 Introduction

Theories and methods of nonnegative polynomials have been widely used in robust control, non-linear control and non-convex optimization [1, 2, 3], etc. Some famous research works on nonnegativity decision of polynomials without cell-decomposition were given by Pólya's Theorem [4, 5] and papers [6, 7].

Recently, Yang [8, 9, 10] introduced a heuristic method for nonnegativity decision of polynomials, which is now called Successive Difference Substitution (SDS). It has been applied to prove a great many polynomial inequalities with more variables and higher degrees. Yao [11] investigated the weighted difference substitutions instead of the original difference substitutions, and proved that, for a form (namely, a homogeneous polynomial) which is positive definite on $\mathbb{R}_+^n$, the corresponding sequence of SDS sets is positively terminating, where $\mathbb{R}_+^n = \{(x_1, x_2, ..., x_n)|x_i \geq 0, i = 1, 2, ..., n\}$. That is, we can decide the nonnegativity of a positive definite form by successively running SDS finite times. The research results above are all confined to the variable substitutions characterized by "difference".

Unlike existing research, this paper regards simplex subdivisions as new perspectives to study variable substitutions, and obtains various effective variable substitutions for deciding positive semi-definite forms and indefinite forms on $\mathbb{R}_+^n$, which are beyond the weighted difference substitutions characterized by "difference". The paper is organized as follows. Section 2 gives some subdivisions of the simplex $\mathbb{T}_n$, and establishes the relationships between them and their corresponding substitutions. Section 3 introduces the concept of the termination of $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$, which is directly related to the positive semi-definite property of the form F. Section 4 proves

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that if the form F is indefinite on $\mathbb{R}_+^n$ and the sequence of the successive L-substitution sets is convergent, then the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating. An algorithm for deciding an indefinite form with a counter-example is presented in Section 5, and by using the obtained algorithm, several examples are listed in Section 6.

2 Simplex subdivisions and the corresponding substitutions

We first introduce some definitions and notations.

Definition 2.1. Let $V = [v_{ij}]$ be an $n \times n$ matrix. If $\sum\limits_{i=1}^{n} v_{ij} = 1, j = 1, 2, \cdots n$, V is called a normalized matrix. And the corresponding linear transformation

$$X^{\mathrm{Tr}} = VT^{\mathrm{Tr}}, \tag{1}$$

is called a normalized substitution, where $X, T \in \mathbb{R}_+^n$, X^{Tr}, T^{Tr} are respectively the transposes of X, T, and V is called the substitution matrix of (1).

Lemma 2.1. Let $U = [u_{ij}] = V_1 V_2 \cdots V_k$. If $V_1, V_2, \cdots, V_k$ are all normalized matrices, then U is a normalized matrix , that is,

$$\sum_{i=1}^{n} u_{ij} = 1 (j = 1, 2, \cdots n).$$

And let $(x_1, x_2, \cdots, x_n)^{\mathrm{Tr}} = U(t_1, t_2, \cdots, t_n)^{\mathrm{Tr}}$, then $\sum\limits_{i=1}^{n} x_i = 1$ iff $\sum\limits_{i=1}^{n} t_i = 1$.

The proof of Lemma 2.1 is very straightforward and is omitted.

Definition 2.2. If the form $F(X) \geq 0$ for all $X \in \mathbb{R}_+^n$, then F is called positive semi-definite on $\mathbb{R}_+^n$, and the set of all positive semi-definite forms is denoted by PSD; If $F(X) > 0$ for all $X(\neq 0) \in \mathbb{R}_+^n$, then F is called positive definite on $\mathbb{R}_+^n$(briefly, a PD); If there are X and $Y \in \mathbb{R}_+^n$, such that $F(X) > 0$ and $F(Y) < 0$, then F is called indefinite on $\mathbb{R}_+^n$.

The $(n-1)$-dimensional simplex is defined as follows

$$\mathbb{T}_n = \{(x_1, x_2, \cdots, x_n) | \sum_{i=1}^{n} x_i = 1, (x_1, x_2, \cdots, x_n) \in \mathbb{R}_+^n\}.$$

Definition 2.3. If the form $F(X) \geq 0$ for all $X \in \mathbb{T}_n$, then F is called positive semi-definite on $\mathbb{T}_n$; If $F(X) > 0$ for all $X \in \mathbb{T}_n$, then F is called positive definite on $\mathbb{T}_n$; If there are X and $Y \in \mathbb{T}_n$ such that $F(X) > 0$ and $F(Y) < 0$, then F is called indefinite on $\mathbb{T}_n$.

It is easy to get the following conclusion.

Lemma 2.2. The form F is positive semi-definite (positive definite, indefinite) on $\mathbb{R}_+^n$ iff F is positive semi-definite (positive definite, indefinite) on $\mathbb{T}_n$.

According to Lemma 2.2, for brevity, we suppose the form F is defined on $\mathbb{T}_n$ in the following sections.

2.1 The barycentric subdivision and the weighted difference substitutions

In this subsection, we focus on the barycentric subdivision and its corresponding substitutions.

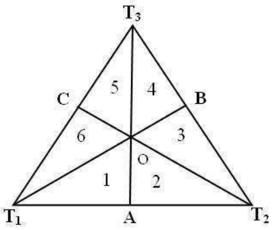


Figure 1: the barycentric subdivision

See Fig.1. The first barycentric subdivision of the simplex $\mathbb{T}_3$ consists of 6 subsimplexes. Consider the subsimplex T_1AO labeled 1 first, where $T_1 = (1,0,0), A = (\frac{1}{2}, \frac{1}{2}, 0), O = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ in the $T_1T_2T_3$-coordinate system. Consruct the following substitution

$$X^{\mathrm{Tr}} = W_3 T^{\mathrm{Tr}}, \tag{2}$$

where

$$W_3 = \begin{bmatrix} T_1 \\ A \\ O \end{bmatrix}^{\mathrm{Tr}} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}. \tag{3}$$

By (2), if $X = T_1 = (1,0,0)$, then $T = (1,0,0)$; If $X = A = (\frac{1}{2}, \frac{1}{2}, 0)$, then $T = (0,1,0)$; If $X = O = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, then $T = (0,0,1)$. And it is indicated that the subsimplex T_1AO and the substitution (2) correspond to each other.

Analogously, the other five subsimplexes labeled as 2-6 correspond to the five substitutions, respectively, whose substitution matrices are as follows

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}. \tag{4}$$

The six substitutions above are just the weighted difference substitutions for $n = 3$, and the set which consists of all the substitutions is called the weighted difference substitution set. Therefore, the fisrt barycentric subdivision of $\mathbb{T}_3$ corresponds to the weighted difference substitution set, that is, there is a one-to-one correspondence between the subsimplexes of the first barycentric subdivision of $\mathbb{T}_3$ and the weighted difference substitutions.

Definition 2.4. Let PW_3 be the weighted difference substitution matrix set for $n = 3$. The set of linear transformations

$$\{X^{\mathrm{Tr}} = B_{[\alpha_1]} B_{[\alpha_2]} \cdots B_{[\alpha_m]} T^{\mathrm{Tr}} | B_{[\alpha_i]} \in PW_3\},$$

is called the m-times successive weighted difference substitution set, which consists of 6^m substitutions.

By Lemma 2.1 and Definition 2.4, we have that the m-th barycentric subdivision of the simplex $\mathbb{T}_3$ corresponds to the m-times successive weighted difference substitution set.

2.2 Some more subdivisions and the corresponding substitutions

Next, we'll present some more subdivisions of the simplex $\mathbb{T}_3$ and their corresponding substitutions.

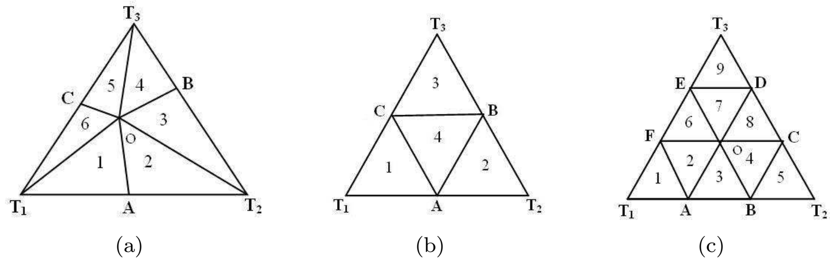


Figure 2: Some subdivisions

Consider Fig.2(a) first. Let $T_1 = (1,0,0)$, $T_2 = (0,1,0)$, $T_3 = (0,0,1)$, $O = (a_0, b_0, c_0)$, $A = (a_1, b_1, 0)$, $B = (0, b_2, c_2)$, $C = (a_3, 0, c_3)$. And the first subdivision of the simplex $\mathbb{T}_3$ consists of six subsimplexes labeled as 1-6, which correspond to the following substitution matrices, respectively.

$$\begin{bmatrix} T_1 \\ A \\ O \end{bmatrix}^{\mathrm{Tr}} = \begin{bmatrix} 1 & a_1 & a_0 \\ 0 & b_1 & b_0 \\ 0 & 0 & c_0 \end{bmatrix}, \begin{bmatrix} T_2 \\ A \\ O \end{bmatrix}^{\mathrm{Tr}} = \begin{bmatrix} 0 & a_1 & a_0 \\ 1 & b_1 & b_0 \\ 0 & 0 & c_0 \end{bmatrix}, \begin{bmatrix} T_2 \\ B \\ O \end{bmatrix}^{\mathrm{Tr}} = \begin{bmatrix} 0 & 0 & a_0 \\ 1 & b_2 & b_0 \\ 0 & c_2 & c_0 \end{bmatrix},$$
$$\begin{bmatrix} T_3 \\ B \\ O \end{bmatrix}^{\mathrm{Tr}} = \begin{bmatrix} 0 & 0 & a_0 \\ 0 & b_2 & b_0 \\ 1 & c_2 & c_0 \end{bmatrix}, \begin{bmatrix} T_3 \\ C \\ O \end{bmatrix}^{\mathrm{Tr}} = \begin{bmatrix} 0 & a_3 & a_0 \\ 0 & 0 & b_0 \\ 1 & c_3 & c_0 \end{bmatrix}, \begin{bmatrix} T_1 \\ C \\ O \end{bmatrix}^{\mathrm{Tr}} = \begin{bmatrix} 1 & a_3 & a_0 \\ 0 & 0 & b_0 \\ 0 & c_3 & c_0 \end{bmatrix}. \tag{5}$$

where $a_i \neq 0, b_i \neq 0$, and $c_i \neq 0$. If $O = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $A = (\frac{1}{2}, \frac{1}{2}, 0)$, $B = (0, \frac{1}{2}, \frac{1}{2})$, $C = (\frac{1}{2}, 0, \frac{1}{2})$, then the subdivision is just the barycentric subdivision.

In Fig.2(b), the first subdivision of the simplex $\mathbb{T}_3$ consists of four subsimplexes labeled as 1-4, which correspond to the following substitution matrices, respectively.

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{6}$$

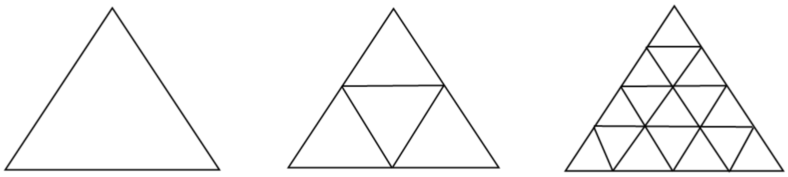
In Fig.2(c), the first subdivision of the simplex $\mathbb{T}_3$ consists of nine subsimplexes labeled as 1-9, which correspond to the following substitution matrices, respectively.

$$\begin{bmatrix} 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix},$$
$$\begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 1 & \frac{2}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ \frac{2}{3} & \frac{2}{3} & 1 \end{bmatrix}. \tag{7}$$

Next, we'll give the definitions of the self-similar subdivision sequence and the successive substitution set.

Definition 2.5. Given a simplex K. Let $\mathrm{sd}^0(K) = K$, and $\mathrm{sd}^i(K) = \mathrm{sd}(\mathrm{sd}^{i-1}(K))$, where $\mathrm{sd}^i(K)$ is the subdivision of $\mathrm{sd}^{i-1}(K)$ for $i = 1, 2, \cdots$. For an arbitrary subsimplex σ of $\mathrm{sd}^i(K)(i = 0, 1, \cdots)$, if the vertex coordinates of all the subsimplexes of $\mathrm{sd}(\sigma)$ in the σ-coordinate system equal to the corresponding ones of all the subsimplexes of $\mathrm{sd}(K)$ in the K-coordinate system, then $\mathrm{sd}^i(K)$ is called the i-th self-similar subdivision of the simplex K, and the subdivision sequence $\{\mathrm{sd}^i(\mathbb{T}_n)\}_{i=1}^{\infty}$ is called self-similar.

Fig.3 shows the first and second self-similar subdivisions of the simplex $\mathbb{T}_3$.



(a) (b) (c)

Figure 3: the self-similar subdivisions of the simplex $\mathbb{T}_3$

Definition 2.6. Let PA be the L-substitution matrix set. The set of linear transformations

$$\{X^{\mathrm{Tr}} = A_{[1]}A_{[2]}\cdots A_{[m]}T^{\mathrm{Tr}} | A_{[i]} \in PA, i = 1, 2, \cdots, m\},$$

is called the m-times successive L-substitution set, briefly, the m-times SLS set.

Suppose that the the first subdivision of the simplex $\mathbb{T}_3$ corresponds to the normalized L-substitution set. By Lamma 2.1, Definition 2.5 and Definition 2.6, we conclude that the m-th self-similar subdivision of the simplex $\mathbb{T}_3$ corresponds to the m-times successive L-substitution set.

Easily, all the subdivisions of the simplex $\mathbb{T}_3$ in the paper can be expanded to the case of the $(n-1)$-dimensional simplex $\mathbb{T}_n$.

2.3 Convergence of the subdivision sequence of a simplex

In this subsection, we'll introduces the concept of the convergence of the subdivision sequence, and presents a sufficient and necessary condition for the convergent self-similar subdivision sequence.

Definition 2.7. Let σ be a subsimplex of $\mathbb{T}_n$, the maximum distance between vertexs of σ is called the diameter of σ.

Definition 2.8. Let K and $\mathrm{sd}^i(K)$ be defined as Definition 2.5. If for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that all the diameters of the subsimplexes of $\mathrm{sd}^N(K)$ are less than ε, the subdivision sequence $\{\mathrm{sd}^i(K)\}_{i=1}^{\infty}$ is called convergent.

Definition 2.9. Suppose that the subdivision scheme through which $\mathrm{sd}^{i-1}(K)$ is subdivided into $\mathrm{sd}^i(K)$ corresponds to the substitution set L_i for $i = 1, 2, \cdots$. If the subdivision sequence $\{\mathrm{sd}^i(K)\}_{i=1}^{\infty}$ is convergent, then the sequence of substitution sets $\{L_i\}_{i=1}^{\infty}$ is called convergent, and if $L_i = L$, $i = 1, 2, \cdots$, briefly, we say that the sequence of the successive L-substitution sets is convergent.

Next, we'll consider the convergence of the sequence of the successive weighted difference substitution sets.

Lemma 2.3. [12, 13] Let K be a complex. If K_N is the k-th barycentric subdivision of K, then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$, such that all the diameters of the subsimplexes of K_N are less than ε.

By Lemma 2.3 , we have the following theorem.

Theorem 2.1. The barycentric subdivision sequence $\{K_i\}_{i=1}^{\infty}$ of $\mathbb{T}_n$ and the corresponding sequence of the successive weighted difference substitution sets are convergent.

Now, we present a sufficient and necessary condition for the convergent self-similar subdivision sequence, which plays important roles for nonnegativity decision of forms.

Theorem 2.2. Let $\{\mathrm{sd}^i(\mathbb{T}_n)\}_{i=1}^{\infty}$ be the self-similar subdivision sequence of $\mathbb{T}_n$, then it is convergent iff an arbitrary 1-dimensional proper face of subsimplexes of $\mathrm{sd}(\mathbb{T}_n)$ is not the 1-dimensional one of $\mathbb{T}_n$.

Proof. The proof of necessity is straightforward by Definition 2.5 and Definition 2.8. Next, we prove sufficiency. Let d be the diameter of $\mathbb{T}_n$, and r be the maximum diameter of subsimplexes of $\mathrm{sd}(\mathbb{T}_n)$, then $r \leq d$. Since an arbitrary 1-dimensional proper face of subsimplexes of $\mathrm{sd}(\mathbb{T}_n)$ is not the 1-dimensional one of $\mathbb{T}_n$, we have $r < d$. Note $l = \frac{r}{d}$, then $l < 1$. Obviously, the maximum diameter of subsimplexes of the N-th self-similar subdivision of $\mathbb{T}_n$ is $l^N d$. When $N \longrightarrow \infty$, then $l^N d \longrightarrow 0$, so the self-similar subdivision sequence $\{\mathrm{sd}^i(\mathbb{T}_n)\}_{i=1}^{\infty}$ is convergent. $\square$

Consider the following subdivisions of $\mathbb{T}_3$ in Fig.4.

In Fig.4(a), the first subdivision of the simplex $\mathbb{T}_3$ consists of 3 subsimplexes labeled as 1-3, which correspond to the following substitution matrices, respectively.

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \\ 1 & 0 & \frac{1}{3} \end{bmatrix}. \tag{8}$$

In Fig.4(b), the first subdivision of the simplex $\mathbb{T}_3$ consists of 5 subsimplexes labeled as 1-5, which correspond to the following substitution matrices, respectively.

$$\begin{bmatrix} 1 & 0 & a_0 \\ 0 & 1 & b_0 \\ 0 & 0 & c_0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a_0 \\ 1 & b_1 & b_0 \\ 0 & c_1 & c_0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & a_0 \\ 0 & b_1 & b_0 \\ 1 & c_1 & c_0 \end{bmatrix}, \begin{bmatrix} 0 & a_2 & a_0 \\ 0 & 0 & b_0 \\ 1 & c_2 & c_0 \end{bmatrix}, \begin{bmatrix} 1 & a_2 & a_0 \\ 0 & 0 & b_0 \\ 0 & c_2 & c_0 \end{bmatrix}. \tag{9}$$

where $O = (a_0, b_0, c_0)$, $A = (a_1, b_1, 0)$, and $B = (0, b_2, c_2)$.

In Fig.4, it is straightforward that the 1-dimensional proper face T_1T_2 of the subsimplex T_1T_2O is just the one of $T_1T_2T_3$. By Theorem 2.2, we have that the two self-similar subdivision sequences of $\mathbb{T}_n$ aren't convergent.

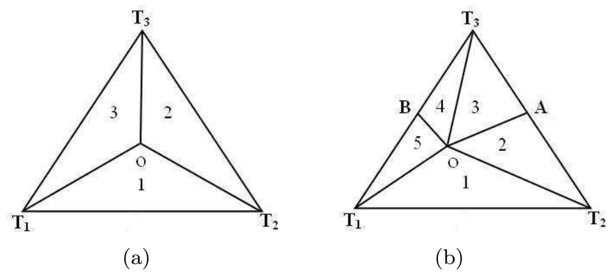


Figure 4: Some more subdivisions

3 Termination of the Sequence of the Successive Substitution Sets of a Form

In this section , we'll define the termination of the sequence of the successive substitution sets of a form, which is directly related to the positive semi-definite property of the form F.

Let $PA = \{A_{[i]} | i = 1, 2, \cdots, k\}$ be the L-substitution matrix set.

Definition 3.1. Given the form F on $\mathbb{T}_n$, when $[a_1], [a_2], \cdots, [a_m]$ traverse all the memebers of $1, 2, \cdots, k$ respectively, we define the set

$$\mathrm{SLS}^{(m)}(F) = \bigcup_{[\alpha_m]}^{k} \cdots \bigcup_{[\alpha_2]}^{k} \bigcup_{[\alpha_1]}^{k} F(A_{[\alpha_1]} A_{[\alpha_2]} \cdots A_{[\alpha_m]} X^{\mathrm{T}}).$$

which is called the m-times successive L-substitution set of the form F.

Definition 3.2. We define the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ as follows

$$\{\mathrm{SLS}(F)^{(m)}\}_{m=1}^{\infty} = \mathrm{SLS}(F), \mathrm{SLS}^{(2)}(F), \cdots.$$

Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \in \mathbb{N}^n$, and let $|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_n$. Then we write a form F with degree d as

$$F = \sum_{|\alpha|=d} c_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

Definition 3.3. The form F is called trivially positive if the coefficients c_α of every term $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ in F are nonnegative; If $F(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}) < 0$, then F is called trivially negative.

Lemma 3.1. Given the form F on $\mathbb{T}_n$, if the form F is trivially positive, then $F \in \mathrm{PSD}$; If the form F is trivially negative, then $F \overline{\in} \mathrm{PSD}$.

Definition 3.4. Given a form F on $\mathbb{T}_n$, if there is a positive integer k such that every element of the set $\mathrm{SLS}^{(k)}(F)$ is trivially positive, then the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is called positively terminating; If there is a positive integer k and a form G such that $G \in \mathrm{SLS}^{(k)}(F)$ and G is trivially negative, the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is called negatively terminating; The sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is neither positively terminating nor negatively terminating, then it is called not terminating.

4 Nonnegativity decision of forms

Given a form F on $\mathbb{T}_n$. We know that if the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating, then we can conclude that the form F is positive semi-definite. Thus there is a natural question, that is, which kind of forms can be solved by the method? The following theorem answers the question.

Theorem 4.1. Let the form F be positive definite on $\mathbb{T}_n$. If the sequence of the successive L-substitution sets is convergent, then the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating.

Proof. We only give the proof for the ternary form with degree d, and the multivariate form can be gotten by induction.

Suppose that $F(x_1, x_2, x_3) = \sum\limits_{i+j+k=d} a_{ijk} x_1^i x_2^j x_3^k$. An arbitrary m-times successive L-substitution can be written as

$$\begin{cases} x_1 = k_{11}u_1 + k_{12}u_2 + k_{13}u_3, \\ x_2 = k_{21}u_1 + k_{22}u_2 + k_{23}u_3, \\ x_3 = k_{31}u_1 + k_{32}u_2 + k_{33}u_3. \end{cases} \tag{10}$$

Let

$$\begin{array}{lll} K_1 = k_{11} + k_{21} + k_{31}, & K_2 = k_{12} + k_{22} + k_{32}, & K_3 = k_{13} + k_{23} + k_{33}, \\ t_1 = K_1 u_1, & t_2 = K_2 u_2, & t_3 = K_3 u_3, \\ k_1 = \frac{k_{11}}{K_1}, & k_2 = \frac{k_{21}}{K_1}, & k_3 = \frac{k_{31}}{K_1}, \\ \alpha_1 = \frac{k_{12}}{K_2} - k_1, & \alpha_2 = \frac{k_{22}}{K_2} - k_2, & \alpha_3 = \frac{k_{32}}{K_2} - k_3, \\ \beta_1 = \frac{k_{13}}{K_3} - k_1, & \beta_2 = \frac{k_{23}}{K_3} - k_2, & \beta_3 = \frac{k_{33}}{K_3} - k_3. \end{array}$$

then (10) becomes

$$\begin{cases} x_1 = k_1 t_1 + (k_1 + \alpha_1) t_2 + (k_1 + \beta_1) t_3, \\ x_2 = k_2 t_1 + (k_2 + \alpha_2) t_2 + (k_2 + \beta_2) t_3, \\ x_3 = k_3 t_1 + (k_3 + \alpha_3) t_2 + (k_3 + \beta_3) t_3, \end{cases} \tag{11}$$

where $\sum\limits_{i=1}^{3} k_i = 1$, $\sum\limits_{i=1}^{3} \alpha_i = 0$ and $\sum\limits_{i=1}^{3} \beta_i = 0$.

Let $t = t_1 + t_2 + t_3$, (11) can be rewritten as

$$\left\{\begin{array}{l} x_1 = k_1 t + \alpha_1 t_2 + \beta_1 t_3, \\ x_2 = k_2 t + \alpha_2 t_2 + \beta_2 t_3, \\ x_3 = k_3 t + \alpha_3 t_2 + \beta_3 t_3. \end{array}\right. \tag{12}$$

Thus

$$\begin{aligned}
\Phi(t_1, t_2, t_3) &= F(k_1 t + \alpha_1 t_2 + \beta_1 t_3, k_2 t + \alpha_2 t_2 + \beta_2 t_3, k_3 t + \alpha_3 t_2 + \beta_3 t_3) \\
&= \sum_{i+j+k=d} a_{ijk} (k_1 t + \alpha_1 t_2 + \beta_1 t_3)^i (k_2 t + \alpha_2 t_2 + \beta_2 t_3)^j (k_3 t + \alpha_3 t_2 + \beta_3 t_3)^k \\
&= \sum_{i+j+k=d} a_{ijk} (k_1^i t^i + \sum_{p+q+r=i, p\neq i} \frac{i!}{p!q!r!} k_1^p \alpha_1^q \beta_1^r t^p t_2^q t_3^r) \cdot \\
&\quad (k_2^j t^j + \sum_{p+q+r=j, p\neq j} \frac{j!}{p!q!r!} k_2^p \alpha_2^q \beta_2^r t^p t_2^q t_3^r)(k_3^i t^i + \sum_{p+q+r=k, p\neq k} \frac{k!}{p!q!r!} k_3^p \alpha_3^q \beta_3^r t^p t_2^q t_3^r) \\
&= \sum_{i+j+k=d} a_{ijk} k_1^i k_2^j k_3^k t^d + \sum_{i+j+k=d} \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) t_1^i t_2^j t_3^k \\
&= F(k_1, k_2, k_3) t^d + \sum_{i+j+k=d} \phi_{ijk}(k_1, k_1, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) t_1^i t_2^j t_3^k \\
&= \sum_{i+j+k=d} \left(\frac{d!}{i!j!k!} F(k_1, k_2, k_3) + \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \right) t_1^i t_2^j t_3^k \\
&= \sum_{i+j+k=d} A_{ijk} t_1^i t_2^j t_3^k \\
&= \sum_{i+j+k=d} A_{ijk} u_1^i u_2^j u_3^k K_1^i K_2^j K_3^k.
\end{aligned} \tag{13}$$

where

$$A_{ijk} = \frac{d!}{i!j!k!} F(k_1, k_2, k_3) + \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3). \tag{14}$$

Obviously,

$$\lim_{(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) \to (0,0,0,0,0,0)} \phi_{ijk}(k_1, k_2, k_3, \alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = 0. \tag{15}$$

And since $F(x_1, x_2, x_3)$ is positive definite on $\mathbb{T}_n$, there exists $\varepsilon > 0$, such that

$$F(k_1, k_2, k_3) \geq \varepsilon > 0. \tag{16}$$

On the one hand, the vertexs of the subsimplex which corresponds to the successive L-substitution (11) are respectively

$$(k_1, k_2, k_3), (k_1 + \alpha_1, k_2 + \alpha_2, k_3 + \alpha_3), (k_1 + \beta_1, k_2 + \beta_2, k_3 + \beta_3).$$

By Theorem (2.1), $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ can be sufficiently small when m is sufficiently large.

On the other hand, for F is continuous on $\mathbb{T}_n$ and by (14)-(16), we have $A_{ijk} > 0$ when $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are sufficiently small.

Putting together the above two aspects, we have that there exists a sufficiently large integer m, such that F becomes trivially positive by (11). For the successive L-substitution (10) is arbitrary, the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is positively terminating. $\square$

Let L be the Yang-Yao's substitution set in Theorem 4.1, then the theorem is the main result in [11]. However, the proof of the theorem in this paper is different from the one given in [11].

According to the proof of Theorem 4.1, we obtain the following conclusion.

Corollary 4.1. Let the form F be positive definite on $\mathbb{T}_n$. If the sequence of the successive L-substitution sets is convergent, then by an arbitrary m-times successive L-substitution, when m is sufficiently larger, F can become a nonlacunary form whose coefficients are all positive.

In Theorem 4.1, if the sequence of the successive L-substitution sets isn't convergent, the conclusion of the theorem is not always true.

Theorem 4.2. Let the form F be positive definite on $\mathbb{T}_n$. If the sequence of the successive L-substitution sets isn't convergent, the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ isn't always negatively terminating.

Proof. Let L be the substitution set, which correspands to the subdivision of the simplex $\mathbb{T}_3$ given by Fig.4(a). And we have concluded that the sequence of the successive L-substitution sets isn't convergent.

Given the form

$$F(x_1, x_2, x_3) = (x_1 - x_2 + x_3)^2 + x_2^2, \quad (x_1, x_2, x_3) \in \mathbb{T}_3. \tag{17}$$

Apparently, F is positive definite on $\mathbb{T}_n$. Consider the m-times successive L-substitution

$$X^{\mathrm{Tr}} = A_{[\alpha]}^m T^{\mathrm{Tr}},$$

where

$$A_{[\alpha]} = \begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix}. \tag{18}$$

And without difficulty, we have

$$\begin{aligned} A_{[\alpha]}^m &= \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3^m} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 0 & \frac{1}{2}(1 - \frac{1}{3^m}) \\ 0 & 1 & \frac{1}{2}(1 - \frac{1}{3^m}) \\ 0 & 0 & \frac{1}{3^m} \end{bmatrix}, \end{aligned} \tag{19}$$

Then

$$\begin{aligned} F(A_{[\alpha]}^m \cdot (x_1, x_2, x_3)^{\mathrm{T}}) =& x_1^2 - 2x_1x_2 + \frac{2}{3^m} x_1 x_3 + 2x_2^2 + (1 - \frac{1}{3^{m-1}}) x_2 x_3 + \\ & \frac{1}{4}(\frac{5}{3^{2m}} - \frac{2}{3^m} + 1) x_3^2. \end{aligned} \tag{20}$$

When $m \longrightarrow \infty$, the coefficient of the term $x_1 x_2$ is -2. Therefore, the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ isn't terminating. □

5 Decision of indefinite forms

Many problems, such as the inequality disproving, are always transformed into decision of indefinite forms. Given a form F on $\mathbb{T}_n$. Suppose that there exists $X_0 \in \mathbb{T}_n$ such that $F(X_0) > 0$. It is well-known to us that if the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating, then the form F is indefinite. Then it follows a question naturally: for a indefinite form F on $\mathbb{T}_n$, is the $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ negatively terminating? The question is answered by the following theorem.

Theorem 5.1. Let the form F be indefinite on $\mathbb{T}_n$. If the sequence of the successive L-substitution sets is convergent, then the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating.

Proof. Let PA be the substitution matrix set which corresponds to the substitution set L. Since the form F is indefinite on $\mathbb{T}_n$, there exists $X_0 \in \mathbb{T}_n$ such that $F(X_0) < 0$. And F is continuous on $\mathbb{T}_n$, so there exists a neighborhood $\mathrm{U}(X_0) \subset \mathbb{T}_n$ of X_0 (If X_0 is on the boundary of $\mathbb{T}_n$, then we take $\mathrm{U}(X_0) \bigcap \mathbb{T}_n$) such that $F(X) < 0$ for all $X \in \mathrm{U}(X_0)$. For the sequence of the successive L-substitution sets is convergent, then there exists a subsimplex σ of the k-th subdivision of $\mathbb{T}_n$, which corresponds to the k-times successive L-substitution

$$X^{\mathrm{Tr}} = B_{[i_1]} B_{[i_2]} \cdots B_{[i_k]} T^{\mathrm{Tr}}, \quad B_{[i_1]}, B_{[i_2]}, \cdots, B_{[i_k]} \in PW_n,$$

satisfying $\sigma \subset \mathrm{U}(X_0)$, where k is a sufficiently larger integer. Thus, $-F(X)$ is positive definite on σ. By Theorem 4.1, the sequence of sets $\{\mathrm{SLS}^{(m)}(-F)\}_{m=1}^{\infty}$ is positively terminating, so there exists a l-times successive L-substitution

$$X^{\mathrm{Tr}} = B_{[j_1]} B_{[j_2]} \cdots B_{[j_l]} T^{\mathrm{Tr}}, \quad B_{[j_1]}, B_{[j_2]}, \cdots, B_{[j_l]} \in PW_n,$$

satisfying that $F(B_{[i_1]} B_{[i_2]} \cdots B_{[i_k]} B_{[j_l]} B_{[j_2]} \cdots B_{[j_l]} T^{\mathrm{Tr}})$ is trivially negative. Therefore, the sequence of sets $\{\mathrm{SLS}^{(m)}(F)\}_{m=1}^{\infty}$ is negatively terminating. □

By the proving process of Theorem (5.1), we obtain the following algorithm, which is used to decide the indefinite form with a counter-example.

Algorithm (SLS)

Input: the form $F \in \mathbb{Q}[x_1, x_2, \cdots, x_n]$, where F is positive definite or indefinite on $\mathbb{R}_+^n$.
Output: "$F \in$PSD", or "$F(\tilde{X}_0) < 0$".
step1: Let $\mathbb{F} = \{F\}$.
step2: Compute $\bigcup\limits_{F \in \mathbb{F}} \mathrm{SLS}(F)$,

Let

$$\mathbb{F} = \bigcup_{F \in \mathbb{F}} \mathrm{SLS}(F) - \{\text{trivially positive forms in} \bigcup_{F \in \mathbb{F}} \mathrm{SLS}(F)\} \triangleq \{F_{[1]}, F_{[2]}, \cdots, F_{[k]}\},$$

where

$$F_{[i]} = F(B_{[i]} X^{\mathrm{Tr}}), \quad B_{[i]} \in PW_n.$$

step3: Let $I = [[1], [2], \cdots, [k]]$.

step31: If $\mathbb{F}$ is null, then output "$F \in$PSD", and terminate.

step32: If there is a trivially negative form $F_{[i]} \in \mathbb{F}$, then output "$F(\tilde{X}_0) < 0$", and terminate, where

$$\tilde{X}_0 = B_{I[i]} (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^{\mathrm{Tr}}, \; B_{I[i]} = B_{[I[i][1]]} B_{[I[i][2]]} \cdots B_{[I[i][m]]},$$

$I[i]$ is the i-th component of I, $I[i][j]$ is the j-th component of $I[i]$, and m is the the total number of the components of $I[i]$.

step33: Else, Compute $\bigcup\limits_{F \in \mathbb{F}} \mathrm{SLS}(F)$. Let

$$\begin{aligned} \mathbb{F} &= \bigcup_{F \in \mathbb{F}} \mathrm{SLS}(F) - \{\text{trivially positive forms in} \bigcup_{F \in \mathbb{F}} \mathrm{SLS}(F)\} \\ &= \{F_{[\mathrm{op}(I[1]),1]}, \cdots, F_{[\mathrm{op}(I[1]),l_1]}, F_{[\mathrm{op}(I[2]),1]}, \cdots, F_{[\mathrm{op}(I[2]),l_2]}, \\ &\cdots, F_{[\mathrm{op}(I[k]),1]}, \cdots, F_{[\mathrm{op}(I[k]),l_k]}\}, \end{aligned}$$

where

$$F_{[\mathrm{op}(I[i]),j]} = F(B_{[\mathrm{op}(L[i]),j]} X^{\mathrm{Tr}}),$$

and $\mathrm{op}(I[i])$ extracts operands from $\mathrm{op}(I[i])$. And let

$$\begin{aligned} L = &[[\mathrm{op}(I[1]), 1], \cdots, [\mathrm{op}(I[1]), l_1], [\mathrm{op}(I[2]), 1], \cdots, [\mathrm{op}(I[2]), l_2], \\ &\cdots, [\mathrm{op}(I[k]), 1], \cdots, [\mathrm{op}(I[k]), l_k]], \end{aligned}$$

then go to step3.

By Algorithm SLS, we design a Maple program called SLS, see Appendix. To the program SLS, there are some positive semi-definite forms making the program do not terminate, that is, we cann't decide these positive semi-definite forms by the method.

6 Examples

In this section, we demonstrate the program SLS with some examples on a computer with Intel(R) Core(TM)2 Duo CPU (E7200 @ 2.53GHz) and 3.25G RAM.

Let L_1, L_2, L_3 be the substitution sets which correspond to the subdivisions of the simplex $\mathbb{T}_3$ given by Fig.1, Fig.2(b) and Fig.2(c), respectively. And let PA_i be the L_i-substitution matrix set for $i = 1, 2, 3$.

Example 1. Show that the following form is positive semi-definite on $\mathbb{R}^3_+$,

$$F(x, y, z) = x(x - y)^5 - y(-z - y)^5 - z(x - z)^5. \tag{21}$$

Utilize the program SLS and execute order SLS$(F, PA_i, [x, y, z])$ for $i = 1, 2, 3$. And the running results are shown in Table 1. So F positive semi-definite on $\mathbb{R}^3_+$.

The substitution set	Running SLS times	Output	CPU time (s)
L_1	3	$F \in$PSD	0.016
L_2	3	$F \in$PSD	0.015
L_3	2	$F \in$PSD	0.032

Table 1.

Example 2. Show that the following form is indefinite on $\mathbb{R}^3_+$,

$$F(x, y, z) = x^4y^2 - 2x^4yz + x^4z^2 + 3x^3y^2z - 2x^3yz^2 - 2x^2y^4 - 2x^2y^3z + x^2y^2z^2 + y^6, \tag{22}$$

Execute order SLS$(F, PA_i, [x, y, z])$ for $i = 1, 2, 3$, and the running results are shown in Table 2. Obviously, $F(0, 1, 0) = 1 > 0$, so the form F is indefinite on $\mathbb{R}^3_+$.

The substitution set	Running SLS times	Output	CPU time (s)
L_1	3	$F(\frac{37}{81}, \frac{91}{324}, \frac{85}{324}) < 0$	0.094
L_2	3	$F(\frac{11}{24}, \frac{1}{3}, \frac{5}{24}) < 0$	0.062
L_3	2	$F(\frac{13}{27}, \frac{7}{27}, \frac{7}{27}) < 0$	0.094

Table 2.

Example 3. Prove or disprove that, for $x \geq 0, y \geq 0, z \geq 0$, and $x + y + z \neq 0$,

$$\frac{2}{3}\left(\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}\right) - \left(\frac{x^6 + y^6 + z^6}{3}\right)^{\frac{1}{6}} \geq 0, \tag{23}$$

Take off denominators of the left polynomial, and denote the new polynomial by F. Execute order SLS$(F, PA_i, [x, y, z])$ for $i = 1, 2, 3$, the running results are shown in Table 3. So the inequality doesn't hold.

The substitution set	Running SLS times	Output	CPU time (s)
L_1	5	$F(\frac{2159}{5832}, \frac{3685}{11664}, \frac{3661}{11664}) < 0$	582.750
L_2	5	$F(\frac{7}{24}, \frac{37}{96}, \frac{31}{96}) < 0$	21.469
L_3	3	$F(\frac{31}{81}, \frac{25}{81}, \frac{25}{81}) < 0$	234.484

Table 3.

The examples above indicate the effectiveness of Algorithm SLS. Thus, we obtain various effective substitutions for deciding positive semi-definite forms and indefinite forms which are beyond Yang's substitutions characterized by "difference".

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Appendix. Maple Program SLS

```
SLS:=proc(poly,A,var)
local f,i,j,k,m,n,p,s,t,F,G,H,M,newvar,st,Var:
uses combinat, LinearAlgebra:
t:=time(): F:=[[poly,[0]]]:n:=nops(var):r=100
Var:=convert(var,Vector):
for i to nops(A) do
    for j to n do
       newvar[i,j]:=op(j,convert(A[i].Var,list)):
    od:
od:
for s to r do
    m:=nops(F): f:=[]:
    for k to m do
        G[k]:=[]:
        for i to nops(A) do
            st:={seq(Var[j]=newvar[i,j],j=1..n)}:
            G[k]:=[op(G[k]),[expand(subs(st,F[k][1])),[op(F[k][2]),i]]]:
        od:
    od:
    F:=[seq(op(G[u]),u=1..nops(F))]:
    for i to nops(F) do
        if max([coeffs(F[i][1])]) < 0 then
           print(F[i][2]):
           M :=IdentityMatrix(n):
           for j from 2 to nops(F[i][2]) do
               M:=M.A[F[i][2][j]]:
           od:
           print(convert(M.Vector[column](n,1/n),list)):
           print(time()-t,`second`):
           return ("The form is indefinite"):
       elif min([coeffs(F[i][1])]) >=0 then
            f:=[op(f),i]:
       fi:
    od:
    if nops(f)>0 then
       F:=subs({seq(F[f[p]]=NULL,p=1..nops(f))},F):
    fi:
    if nops(F)=0 then
       print(s):print(time()-t,`second`):
       return("The form is positive semi-definite"):
    fi:
od:
end proc:
```